Journal of Engineering Mathematics 21: 33–40 (1987)

Quasi-linear electrical potentials in steady-state Joule heating

J.H. Young
Department of Physics, University of Alabama at Birmingham, Birmingham, AL 35294, USA

Received 18 July 1986; accepted 13 October 1986

Abstract

It is shown that, when account is taken of the temperature dependence of the electrical conductivity of a medium in steady-state conditions, the electrical potential obeys a quasi-linear, second-order partial differential equation. The equation is shown to be easily solved by means of a generalized Kirchhoff transformation, giving coordinate-free solutions in terms of functions which obey Laplace's equation. Steady-state temperatures resulting from the potential are shown to be significantly influenced by the quasi-linear potential when compared with the expected form which assumes the potential to satisfy Laplace's equation.

1. Introduction

Engineering problems involving theoretical analyses of material temperatures produced by electrical current flow through various types of contacts generally encounter the problem of determining the form of the potential responsible for those current distributions. (See for instance Bryant and Burton [3], Bryant and Oh [4] and Chen and Burton [5]). The usual assumption then made is that the potential can be obtained as a solution to Laplace's equation. It has recently been pointed out [8], however, that when the temperature dependence of the electrical conductivity is taken into account, the potential will not obey the simple linear equation of Laplace. In fact it was shown there that within a typical metal under steady-state conditions, which would be expected if surface cooling were provided, the potential will obey a second-order quasi-linear partial differential equation. The remarkable feature of this equation is the fact that it is easily linearized and yields solutions for the potential in terms of functions which do satisfy Laplace's equation.

The present work has the following purposes: (1) to show that the "quasi-linear potential" is but a member of a class of solutions obtainable by what amounts to a generalized Kirchhoff transformation; (2) to indicate the significance of the quasi-linear potential by illustrating its difference from the ("linear") potential which does satisfy Laplace's equation; (3) to illustrate the differences in the resulting steady-state temperature due to Joule heating according to whether the potential is assumed to satisfy Laplace's equation, corresponding to constant conductivities, or the quasi-linear equation, corresponding to a temperature-dependent electrical conductivity.

2. The quasi-linear equation

A steady-state current density J has, by definition, zero divergence. Within an Ohmic medium having electrical conductivity σ the current density can be expressed in terms of the gradient of the electrical potential, Φ, as

$$J = -\sigma \nabla \Phi. \tag{1}$$

The condition that $\nabla \cdot J = 0$ immediately gives the result that Φ satisfies Laplace's equation *provided* that σ is constant.

Electrical conductivities, however, vary approximately inversely with temperature and the inclusion of conductivity changes with an appreciable temperature variation leads to quite a different picture. For a metal, the electrical and thermal conductivities (κ) depend on (absolute) temperature in a manner described by the Weidemann-Franz law,

$$\frac{\kappa}{\sigma} = \alpha T, \tag{2}$$

where the Lorenz number, α, has the value 2.445×10^{-8} WΩ/°K [7].

The condition that J have zero divergence gives, for variable σ,

$$\nabla^2 \Phi + (1/\sigma) \nabla \sigma \cdot \nabla T = 0. \tag{3}$$

The electrical conductivity gradient can be expressed in terms of a temperature gradient which gives, upon using (2) along with the fact that κ is essentially constant for a typical metal above room temperature,

$$\nabla^2 \Phi - (1/T) \nabla T \cdot \nabla T = 0. \tag{4}$$

It was shown in [8] that the steady-state temperature in the case of applicability of the Weidmann-Franz law can be functionally related to the potential as

$$T(\Phi) = \frac{1}{\sqrt{\alpha}} \left[C - (\Phi - C')^2 \right]^{1/2}, \tag{5}$$

wherein C and C' are integration constants. It is noteworthy that (5) depends only on the *ratio* of thermal to electrical conductivity; it does not depend on either conductivity individually. Use of this result allows (4) to be expressed as

$$\nabla^2 \Phi + \frac{\Phi - C'}{C - (\Phi - C')^2} (\nabla \Phi)^2 = 0. \tag{6}$$

Thus, when account is taken of the fact that the electrical conductivity is (inversely) dependent upon temperature while the thermal conductivity is, at least to good approximation, constant the potential responsible for maintaining steady-state currents is seen to obey a quasi-linear equation. Formidable through (6) might appear to be, it is shown in the next section that a completely general, coordinate-free solution exists for any dimensionality. That solution, when used in (5), will then give the steady-state temperature solution.

3. The quasi-linear solution

The form of (6) can be simplified by a rescaling of the potential. Letting

$$\phi \equiv \frac{1}{\sqrt{C}} (\Phi - C') \tag{7}$$

transforms (6) to

$$\nabla^2 \phi + \left[\frac{\phi}{1 - \phi^2} \right] (\nabla \phi)^2 = 0. \tag{8}$$

When the transformation defined in (7) is used in the $T(\Phi)$ expression the latter becomes

$$T(\phi) = \left(\frac{C}{\alpha}\right)^{1/2} \left[1 - \phi^2\right]^{1/2}, \tag{9}$$

which makes it apparent that $|\phi| \leqslant 1$ in order that T be real.

Equation (8) is of the general form

$$\nabla^2\phi + f(\phi)(\nabla\phi)^2 = 0. \tag{10}$$

This equation is similar to a form discussed by Ames [2] and can easily be solved as follows. Introduce an auxiliary potential function, χ, as

$$\nabla\chi \equiv \nabla\phi \exp\left\{\int f(\phi)\,\mathrm{d}\phi\right\}. \tag{11}$$

It follows, by virtue of ϕ satisfying (10), that

$$\nabla^2\chi = 0, \tag{12}$$

and is thus a function that is readily determined for any desired geometry. Equation (11) is equivalent to the integral relation

$$\chi = \int \exp\left\{\int f(\phi)\,\mathrm{d}\phi\right\}\,\mathrm{d}\phi. \tag{13}$$

For any $f(\phi)$, (10) has thus been solved by integration of (13); its subsequent inversion then gives ϕ in terms of a function (χ) which satisfies Laplace's equation.

The transformation given by (11) actually amounts to nothing more than a generalization of the Kirchhoff transformation utilized in heat-conduction problems in media in which the thermal conductivity is temperature dependent [1]. If the function $f(\phi)$ can be expressed as the logarithmic derivative, of some function $\Lambda(\phi)$,

$$f(\phi) \equiv \frac{1}{\Lambda(\phi)}\frac{\mathrm{d}\Lambda(\phi)}{\mathrm{d}\phi}, \tag{14}$$

equation (11) then gives the ϕ-χ transformation as

$$\nabla\chi = \Lambda(\phi)\nabla\phi. \tag{15}$$

This is immediately recognized as the familiar Kirchhoff transformation.

In the case of present interest $f(\phi) = \phi/(1 - \phi^2)$. Substitution into (13) and inversion of the subsequent integration yields

$$\phi = \sin\chi. \tag{16}$$

Finally, making use of the ϕ-Φ relationship defined in (7) gives

$$\Phi = C' + C^{1/2}\sin\chi. \tag{17}$$

The quasi-linear equation for Φ, namely (6), has thus been solved in general, coordinate-free form applicable to any geometry. Use of this result in (5) then gives the general solution to the steady-state temperature produced by the potential Φ:

$$T=\left(\frac{C}{\alpha}\right)^{1/2}\cos\chi. \tag{18}$$

4. An example

Appreciation of the quasi-linear potential and its significance might best be achieved by means of the following illustration. Suppose that a solid electrical conducting sphere of radius R is capped by a pair of electrodes which hold its northern and southern hemispheres at the constant potentials $+V_0$ and $-V_0$, respectively; that is

$$\Phi(R,\theta)=+V_0,\quad 0<\theta<\frac{\pi}{2};\quad \Phi(R,\theta)=-V_0,\quad \frac{\pi}{2}<\theta<\pi. \tag{19}$$

Electrical currents will then arise throughout the interior of the sphere due to the potential within that region and cause its Joule heating. It will be supposed that the surface of the sphere is maintained at the constant temperature T_0 so that a steady-state temperature distribution is eventually attained.

Making use of the general solution given in (5) and requiring that the temperature be T_0 over the entire surface of the sphere gives

$$T_0=\frac{1}{\sqrt{\alpha}}\left[C-(+V_0-C')^2\right]^{1/2} \tag{20a}$$

and

$$T_0=\frac{1}{\sqrt{\alpha}}\left[C-(-V_0-C')^2\right]^{1/2}. \tag{20b}$$

These enable C and C' to be evaluated, with the results

$$C=\alpha T_0^2+V_0^2;\quad C'=0. \tag{21}$$

Use of these values in (17) gives for the potential

$$\Phi(r,\theta)=\left[\alpha T_0^2+V_0^2\right]^{1/2}\sin\chi(r,\theta). \tag{22}$$

The function $\chi(r,\theta)$ must now be determined so that $\Phi(r,\theta)$ satisfies the conditions stated in (19). It is evident that χ is here an interior solution to Laplace's equation and, in order that the requirement that Φ be an odd function of $\cos\theta$ be met, must be an odd function of $\cos\theta$. Its most general form is then

$$\chi(r,\theta)=\sum_{l=0}^{\infty}a_{2l+1}r^{2l+1}P_{2l+1}(\cos\theta), \tag{23}$$

where $P_l(\cos\theta)$ is the Legendre polynomial of degree l. Use of (23) in (22) and imposing the condition $\Phi(R,\theta)=+V_0$ for $0<\theta<\pi/2$ gives

$$\sum_{l=0}^{\infty}a_{2l+1}R^{2l+1}P_{2l+1}(\cos\theta)=\sin^{-1}\left[\frac{V_0}{\left(\alpha T_0^2+V_0^2\right)^{1/2}}\right]. \tag{24}$$

The series coefficients can be evaluated upon multiplying by $P_{2l'+1}(\mu)$ and integrating over $\mu \equiv \cos\theta$ from 0 to 1:

$$\sum_{l=0}^{\infty} a_{2l+1} R^{2l+1} \int_0^1 P_{2l+1}(\mu) P_{2l'+1}(\mu)\, \mathrm{d}\mu = \sin^{-1}\left[\frac{V_0}{\left(\alpha T_0^2 + V_0^2\right)^{1/2}}\right] \int_0^1 P_{2l'+1}(\mu)\, \mathrm{d}\mu. \quad (25)$$

The integrals are given by Gradshteyn and Ryzhik [6]; these are

$$\int_0^1 P_{2l+1}(\mu) P_{2l'+1}(\mu)\, \mathrm{d}\mu = \frac{\delta_{ll'}}{4l+3}, \quad (26a)$$

$$\int_0^1 P_{2l+1}(\mu)\, \mathrm{d}\mu = \frac{(-1)^l \Gamma(l+1/2)}{2\Gamma(1/2)\Gamma(l+2)}. \quad (26b)$$

The orthogonality indicated in the first of these allows the determination of the a_{2l+1}, giving the result

$$\Phi(r, \theta) = \left[\alpha T_0^2 + V_0^2\right]^{1/2} \sin\left\{\sin^{-1}\left[\frac{V_0}{\left(\alpha T_0^2 + V_0^2\right)^{1/2}}\right] \times \sum_{l=0}^{\infty} \frac{(-1)^l (4l+3)\Gamma(l+1/2)}{2\Gamma(1/2)\Gamma(l+2)} \left(\frac{r}{R}\right)^{2l+1} P_{2l+1}(\cos\theta)\right\}. \quad (27)$$

It has been emphasized here that the temperature dependence of the electrical conductivity mandates that steady-state current distributions are defined by a quasi-linear potential, of which the expression given in (27) is an example. It would be expected that in the case of only very modest temperature elevations attributed to small values of impressed potential, V_0, that σ may, to good approximation be regarded as constant. Expansion of (27) for small V_0 immediately yields the result

$$\Phi(r, \theta) = V_0 \chi(r, \theta), \quad (28)$$

where $\chi(r, \theta)$ is the infinite series appearing in (27). This result says that for small V_0 the potential will indeed be simply a solution to Laplace's equation. Such a potential will hereafter be designated by Φ_L in order to distinguish it from the quasi-linear potential discussed here. (In the other extreme of very large V_0 the inverse sine function appearing in the argument in (27) tends toward the value $\pi/2$).

Figure 1 displays the differences between the quasi-linear potential and Φ_L along the direction $\theta = 0°$ for values of V_0 chosen as 40, 80 and 120 mV. The curves indicate that the quasi-linear potential is significantly different from Φ_L at all points except, of course, at the origin and on the surface. The magnitude of the differences is, as would be expected, clearly dependent upon the size of V_0. Values in the southern hemisphere ($\theta = 180°$) would be just the mirror images of the curves shown in Fig. 1.

The form of the steady-state temperature distribution produced by Joule heating is dependent upon the basic equation obeyed by the potential, i.e., the quasi-linear or Laplace equation. In the former case $T(\Phi)$ is given by (5), or equivalently, by (18). The temperature distribution

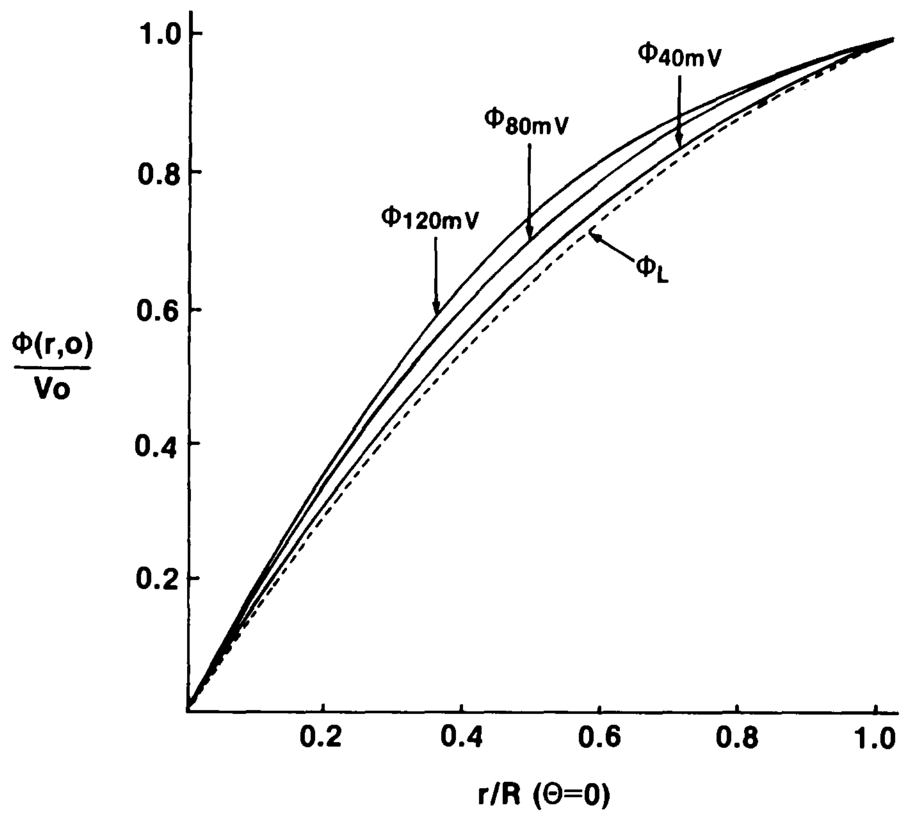

Fig. 1. Ratio of the potential to the impressed potential along the $\theta = 0$ direction. Solid curves give the ratio for the quasilinear potential, while the dashed curve is the Laplace potential. Values of V_0 are 40, 80 and 120 mV.

throughout the above sphere, whose electrical conductivity is inversely dependent upon temperature, is given by

$$T(r, \theta) = \frac{1}{\sqrt{\alpha}}\left[\alpha T_0^2 + V_0^2\right]^{1/2} \cos\left\{\sin^{-1}\left[\frac{V_0}{\left(\alpha T_0^2 + V_0^2\right)^{1/2}}\right]\right. \\ \left. \times \sum_{l=0}^{\infty} \frac{(-1)^l (4l+3)\Gamma(l+1/2)}{2\Gamma(1/2)\Gamma(l+2)}\left(\frac{r}{R}\right)^{2l+1} P_{2l+1}(\cos\theta)\right\}. \tag{29}$$

The case of constant electrical conductivity ($\equiv \sigma_0$), along with the assumption of constant thermal conductivity ($\equiv \kappa_0$) has also been considered in [8]. It has been shown there that the steady-state temperature in that case is given by the quadratic form

$$T(\Phi_L) = \tfrac{1}{2}(\sigma_0/\kappa_0)\left[C - (\Phi_L - C')^2\right], \tag{30}$$

where again C and C' are integration constants. If the conductivities in the sphere example are

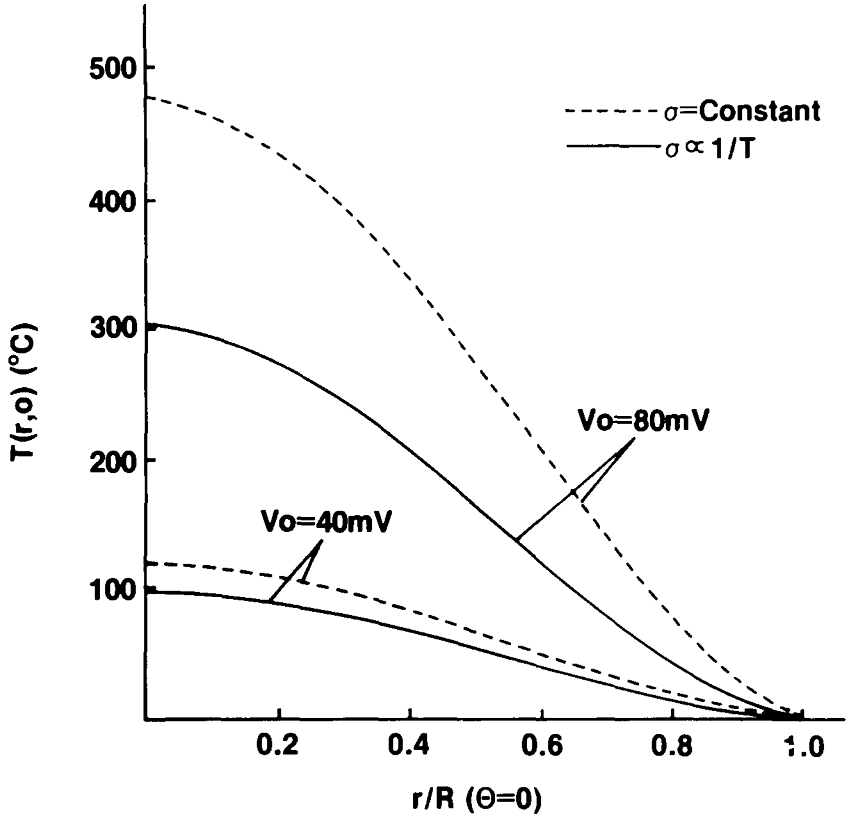

Fig. 2. Steady-state temperatures along the $\theta = 0$ direction. The solid curves plot the temperature for the temperature-dependent conductivity, while the dashed curves assume constant conductivities. Values of V_0 are 40 and 80 mV.

assumed constant, C and C' can again be determined by the requirement of the surface temperature being T_0. Use of (28) for Φ_L then gives

$$T(r,\theta) = T_0 + \frac{V_0^2}{2\alpha T_0}\left\{1 - \left[\sum_{l=0}^{\infty} \frac{(-1)^l(4l+3)\Gamma(l+1/2)}{2\Gamma(1/2)\Gamma(l+2)}\left(\frac{r}{R}\right)^{2l+1} P_{2l+1}(\cos\theta)\right]^2\right\}. \quad (31)$$

wherein $\sigma_0/\kappa_0 = 1/\alpha T_0$ has been used.

It is worthy of mention at this point that the steady-state temperature solution given by equation (12) in the work of Chen and Burton [5] is a member of the class of solutions given here in (30). The direct use of (30) of the present work offers the obvious advantage that it is only necessary to input the appropriate potential function, regardless of its dimensionality or geometry. The same is true, of course, of the more realistic class of temperature solutions given in (5).

Figure 2 compares the temperature solutions given in (29) and (31) for impressed potentials of 40 and 80 mV. T_0 has been chosen at 273° K and the plots are represented on the Celsius scale. The dashed curves, representing constant conductivity values, are seen to deviate from the solid curves, which incorporate the Weidemann-Franz behavior, increasingly as the impressed potential is made larger. The inverse temperature dependence of the electrical conductivity is the factor responsible for keeping the solid curves lower.

5. Conclusion

It has been seen that taking into account the more realistic conductivity behavior of a metal leads to more complicated mathematical considerations, but that these are easily managed. The solution to the quasi-linear potential equation (6) for the case of constant thermal conductivity, has been shown to be given quite generally by (17). The latter solution, derived by means of a generalized Kirchhoff transformation, is valid for any geometry and dimensionality. (Although that solution was presented in [8] and illustrated in one dimension, its complete derivation has only more recently been realized). The differences between the quasi-linear and Laplace potentials are evident and the former should be used whenever substantial temperature changes are anticipated. The steady-state temperatures depend strongly on the form of the applicable potential and highly erroneous results might arise in an analysis which fails to take this point into account.

References

1. W.F. Ames: *Nonlinear Partial Differential Equations in Engineering*, Academic Press (1965) p. 21.
2. W.F. Ames: *Nonlinear Partial Differential Equations in Engineering, Vol. II*, Academic Press (1972) p. 8
3. M.D. Bryant and R.A. Burton: Thermal deformation in brushes for current passage through a steadily moving patch, *Wear* 79 (1982) 83–92.
4. M.D. Bryant and S. Oh: The transient temperature field existing about a cooled stationary electrical contact, *Applied Scientific Research* 42 (1985) 89–106.
5. C.P. Chen and R.A. Burton: Thermoelastic effects in brushes with high current and high sliding speeds, *Wear* 59 (1980) 277–288.
6. I.S. Gradshteyn and I.M. Ryzhik: *Tables of Integrals. Series and Products*, Academic Press (1980).
7. G.T. Meaden: *Electrical Resistance of Metals*, Heywood (1966) p. 71.
8. J.H. Young: Steady state Joule heating with temperature dependent conductivities, *Applied Scientific Research* 43 (1986) 55–65.